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Boundary Condition Effects in Free Vibrations of Higher-Order Soft Sandwich Beams

Vladimir S. Sokolinsky* and Steven R. Nutt†
University of Southern California, Los Angeles, California 90089-0241
and
Yeoshua Frostig‡
Technion—Israel Institute of Technology, 32000 Haifa, Israel

Natural motions of sandwich beams with a transversely flexible- (soft-) core are analyzed based on a higher-order theory formulation. The theory does not resort to the use of presumed displacement patterns and permits imposition of the different support conditions at the same boundary section. Finite differences are used to approximate the governing equations, and the deflated iterative Arnoldi algorithm is applied to solve the algebraic eigenvalue problem. Free vibration predictions of the higher-order theory are shown to be in good agreement with experiments reported in the literature. The face sheet deflections of the sandwich beams with nonidentical support conditions at the same boundary are different in the close vicinity of that boundary. The interaction between the face sheets and the core plays a crucial role in the vibration response of sandwich beams with a soft core. The parametric study shows that the qualitative sequence of the antisymmetric (global) and symmetric (local) vibration modes varies with face sheet thickness and that there are sandwich beam layouts for which some higher vibration modes arise from the interaction of the basic modes.

Nomenclature

b	=	width of sandwich beam
c	=	height of the core
d	=	distance between the centroids of the face sheets
d_b	=	thickness of the lower face sheet
d_t	=	thickness of the upper face sheet
EA_b	=	axial rigidity of the lower face sheet
EA_t	=	axial rigidity of the upper face sheet
E_b	=	Young's modulus of the lower face sheet
E_c	=	Young's modulus of the core
E_t	=	Young's modulus of the upper face sheet
EI_b	=	flexural rigidity of the lower face sheet
EI_t	=	flexural rigidity of the upper face sheet
f	=	frequency, $\omega/(2\pi)$
G_c	=	shear modulus of the core
h	=	height of sandwich beam
i	=	imaginary unit
$\boldsymbol{K}$	=	$n \times n$ finite difference matrix
L	=	length of sandwich beam
$\boldsymbol{M}$	=	$n \times n$ finite difference mass matrix
m_b	=	mass per unit length of the lower face sheet
m_c	=	mass per unit length of the core
m_t	=	mass per unit length of the upper face sheet
t	=	time coordinate
$u_c(x_c, z_c, t)$	=	longitudinal displacement of a point within the core
$u_{0b}(x_b, t)$	=	longitudinal displacement of the centroid line of the lower face sheet
$u_{0t}(x_t, t)$	=	longitudinal displacement of the centroid line of the upper face sheet
$\boldsymbol{u}(x)$	=	5×1 vector of the one-dimensional unknowns $\bar{u}_{0b}(x)$, $\bar{u}_{0t}(x)$, $\bar{w}_b(x)$, $\bar{w}_t(x)$, and $\bar{\tau}(x)$
$\boldsymbol{u}(x, t)$	=	5×1 vector of the main unknowns u_{0b}, u_{0t}, w_b, w_t, and τ
$\boldsymbol{v}$	=	$n \times 1$ finite difference vector of unknowns
$w_b(x_b, t)$	=	vertical displacement of the centroid line of the lower face sheet
$w_{b,x}(x_b, t)$	=	rotation of the centroid line of the lower face sheet
$w_c(x_c, z_c, t)$	=	vertical displacement of a point within the core
$w_t(x_t, t)$	=	vertical displacement of the centroid line of the upper face sheet
$w_{t,x}(x_t, t)$	=	rotation of the centroid line of the upper face sheet
x_b, y_b, z_b	=	local rectangular coordinates of the lower face sheet
x_c, y_c, z_c	=	local rectangular coordinates of the core
x_t, y_t, z_t	=	local rectangular coordinates of the upper face sheet
ρ_b	=	material density of the lower face sheet
ρ_c	=	material density of the core
ρ_t	=	material density of the upper face sheet
$\sigma_{zz}(x_c, z_c, t)$	=	vertical normal stress in the core
ω	=	circular frequency
$\bar{\omega}$	=	nondimensional frequency parameter

Subscript

x	=	differentiation with respect to x_b, x_t, or x_c, when preceded by comma

Superscript

$\ddot{}$	=	second derivative with respect to time

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*Research Scholar, Center for Composite Materials.
†M.C. Gill Professor, Director of Center for Composite Materials, Materials Science Department.
‡Associate Professor, Faculty of Civil Engineering.

I. Introduction

THE increasing use of sandwich construction and the need to predict accurately their vibrational behavior requires more sophisticated modeling approaches that do not contain restrictive assumptions on the mode shapes and boundary conditions.

At present, the analytical models of sandwich structures are largely based on one of the following approaches: classical sandwich theory,[1] elastic foundation model,[2] various higher-order theories, where the higher-order terms are defined at the neutral axis,[3] and, most recently, the higher-order sandwich panel theory (HSAPT).[4,5]

Both experimental and analytical studies[6,7] have clearly demonstrated the high accuracy of the HSAPT-predicted stress and deformation fields, including those induced by concentrated loads or point supports.

One objective of this paper is to use HSAPT to study the dynamic response of soft-core sandwich structures, considering the influence of cross-sectional characteristics. Another is to examine the effects of independently prescribed boundary conditions at the edges of each face sheet, that is, at the same section. This fidelity allows much closer simulation of supports used in actual practice, and the strong influence that boundary conditions have on the sandwich response is shown. The analysis does not resort to presumed vibrational modes and accounts for the interaction between the two face sheets and the soft core. These important distinctive features of the HSAPT provide physical insight into the complex dynamic response of soft-core sandwich structures that cannot be achieved using the first three modeling approaches listed.

A review of selected works chronicles the development of current theoretical understanding. Natural motions of sandwich beams have been studied assuming an incompressible and antiplane core hypothesis.[1,2] The natural motions of fixed–fixed sandwich beams were analyzed using an energy approach in conjunction with the Lagrangian multipliers and were measured by Raville et al.[8] It was concluded that, for the range of frequencies ~100–1200 Hz, the assumptions of infinite modulus of elasticity of the core (in the vertical direction) and negligible rotatory inertia, which were adopted in the paper, were justified. The bending vibrations of cantilever sandwich beams were studied by Markuš and Valášková[9] using a sixth-order differential equation in terms of the transverse displacement of the sandwich beam. It was shown that the location of the nodal points along the span of the sandwich beam was not constant for the particular vibration mode, as it is for the Euler–Bernoulli beam, but depended on the cross section characteristics of the sandwich beam. Silverman[10] modeled a sandwich beam as a thin disk beam-shaped body in plane stress. The longitudinal displacements of the beam were taken to be cubic in the vertical coordinate, whereas the vertical displacements were assumed to be constant through the beam thickness. It was shown that an increase in the thickness of the face sheets does not necessarily lead to an increase in the natural frequency of the sandwich beam irrespective of the mode. The author claimed that the same trend was indicated in the experiments of Cleary and Leadbetter.[11]

Three higher-order refined displacement models with transverse inextensibility for the free vibration analysis of sandwich and composite beams, which avoid the use of a shear correction coefficient (see Ref. 12), were proposed by Marur and Kant.[13] The authors performed numerical experiments for various layouts, boundary conditions, and aspect ratios to compare their higher-order models with the first-order shear deformation theory of Timoshenko (see Refs. 3 and 12), as well as with the results obtained by alternative approaches. Maheri and Adams[14] tried to adopt the Timoshenko beam theory (see Ref. 12) for the analysis of sandwich beams with a stiff core. The authors proposed to choose the shear shape correction factor to fit the experimental data at the high frequency. The results of the analysis based on that experimentally chosen factor were reported to be in a good agreement with the experiment for the free–free sandwich beam. A Green function approach was used by Sakiyama et al.[15] to analyze free vibrations of sandwich beams with elastic and linearly viscoelastic cores and arbitrary boundary conditions, neglecting vertical normal strains in the core. They also presented extensive numerical data concerning natural frequencies and loss factors of the sandwich beams.

In the references cited, efforts were made to account for the nonlinear longitudinal displacements of the core. All of the authors, however, assumed that the elastic modulus of the core in the vertical direction was infinite (incompressible core). This approach could be adopted for the free vibration analysis of stiff-core sandwich beams. However, the response of the sandwich constructions with a soft core should be simulated with the aid of an enhanced theory that can account for its vertical flexibility. This vertical flexibility affects stress and displacement fields in the face sheets and leads to nonlinear longitudinal and vertical displacement patterns in the core.[4] Thus, the classical hypotheses assuming that the cross sections of the core remain planar after deformation and that the height of the beam remains unchanged are no longer valid.

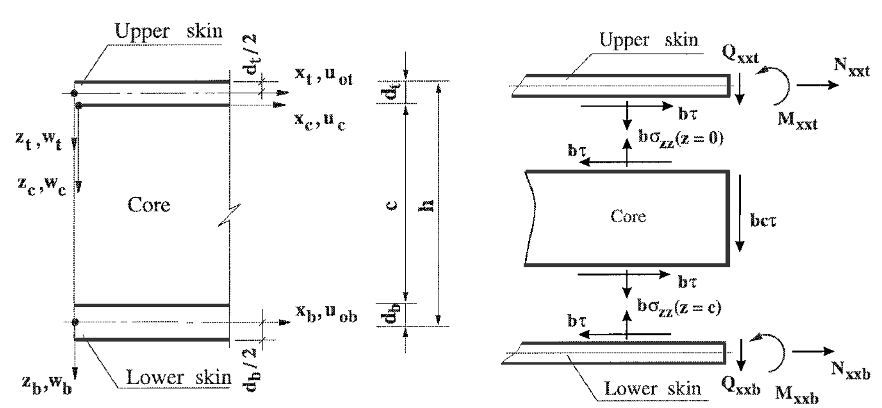
Jensen and Irgens[16] derived the equations of motion for sandwich beams and plates with a soft core and experimentally verified their predictions for the free vibration frequencies and modes of the simply supported symmetric sandwich beam using a television-holography technique. They modeled the soft core as a two-parameter linear elastic and isotropic foundation with shear interaction and normal stress effects[2] (the Vlasov foundation) and the face sheets as ordinary (Euler–Bernoulli) plates or beams. For a simply supported beam with symmetrical cross section and identical boundary conditions at its upper and lower face sheets, which prevent longitudinal movement of the beam, the agreement between the theory and the experiment reported in Ref. 16 was good. Frostig and Baruch[5] derived the free vibration equations for sandwich beams with a transversely flexible core using HSAPT,[4] whose main advantages over the existing analytical approaches were mentioned earlier.

In what follows, the assumptions used by the HSAPT, as well as the equations of motion for the sandwich beam with a soft core and appropriate boundary conditions, are outlined first. Next, quantitative comparisons between the predictions of the vibration response are compared with published experimental results for both soft- and stiff-core sandwich beams. A numerical study of the natural motions of sandwich beams with various boundary conditions is presented. Finally, the effects of cross-sectional characteristics on the free vibration response of the sandwich beams are discussed and conclusions are drawn.

II. Formulation

The assumptions of HSAPT developed in Ref. 4 are as follows (Fig. 1):

1) The face sheets of the sandwich beam, either metallic or symmetric composite laminates, are modeled as ordinary beams with negligible shear strains that follow Euler–Bernoulli assumptions and are subjected to small deformations.



a) Geometry **b) Internal stresses and stress resultants**

Fig. 1 Sandwich beam conventions.

2) The transversely flexible core layer is considered as a two-dimensional elastic medium with small deformations where its height may change under loading, and its cross section does not remain planar. The longitudinal (in-plane) stresses in the core are neglected.

3) The interface layers between the face sheets and the core are assumed to be infinitely rigid and to provide perfect continuity of the deformations at the interfaces.

The justification for the neglect of the in-plane stresses in the soft core has been presented in both theoretical and experimental studies.[4] This hypothesis is widely accepted even for the analysis of sandwich structures with stiff honeycomb cores.[1,2,9]

A. Equations of Motion

The equations describing the free vibrations of sandwich beams with a soft core have been derived in Ref. 5 via the application of Hamilton's variational principle (see Ref. 17). In deriving those equations, it was assumed that the acceleration fields of the face sheets have the same shape as their static deformation fields, and that the effect of the rotatory inertia[12] of the face sheets is negligible. The acceleration field of the core in the longitudinal and vertical directions was assumed to vary linearly with its height. This simplification was made only for modeling the dynamic effects in the core because, in the case of free vibrations, the inertia loads exerted on a sandwich beam are uniformly distributed along the structure. In such circumstances, the nonlinearities arising in the vicinity of supporting points are small and can be disregarded without loss of accuracy. On the other hand, the displacement field in the core was assumed to be nonlinear, both in the longitudinal and vertical directions, in accordance with the higher-order theory assumptions. Confirmation of this statement is given later through quantitative comparisons of predictions of the vibration response produced by the HSAPT with published experimental results.

The governing equations describing the free vibrations of sandwich beams with a soft core and isotropic face sheets are

$$EA_t u_{0t,xx} + b\tau - m_t \ddot{u}_{0t} - \tfrac{1}{3}m_c\ddot{u}_{0t} - \tfrac{1}{6}m_c\ddot{u}_{0b} + \tfrac{1}{6}m_c d_t \ddot{w}_{t,x} - \tfrac{1}{12}m_c d_b \ddot{w}_{b,x} = 0 \tag{1}$$

$$EA_b u_{0b,xx} - b\tau - m_b \ddot{u}_{0b} - \tfrac{1}{6}m_c\ddot{u}_{0t} - \tfrac{1}{3}m_c\ddot{u}_{0b} + \tfrac{1}{12}m_c d_t \ddot{w}_{t,x} - \tfrac{1}{6}m_c d_b \ddot{w}_{b,x} = 0 \tag{2}$$

$$EI_t w_{t,xxxx} - \tfrac{1}{2}b(c+d_t)\tau_{,x} - (bE_c/c)(w_b - w_t) + m_t\ddot{w}_t + \tfrac{1}{6}m_c d_t\ddot{u}_{0t,x} + \tfrac{1}{12}m_c d_t \ddot{u}_{0b,x} - \tfrac{1}{12}m_c d_t^2 \ddot{w}_{t,xx} + \tfrac{1}{24}m_c d_t d_b \ddot{w}_{b,xx} + \tfrac{1}{3}m_c\ddot{w}_t + \tfrac{1}{6}m_c\ddot{w}_b = 0 \tag{3}$$

$$EI_b w_{b,xxxx} - \tfrac{1}{2}b(c+d_b)\tau_{,x} + (bE_c/c)(w_b - w_t) + m_b\ddot{w}_b - \tfrac{1}{12}m_c d_b\ddot{u}_{0t,x} - \tfrac{1}{6}m_c d_b \ddot{u}_{0b,x} - \tfrac{1}{12}m_c d_b^2 \ddot{w}_{b,xx} + \tfrac{1}{24}m_c d_t d_b \ddot{w}_{t,xx} + \tfrac{1}{3}m_c\ddot{w}_b + \tfrac{1}{6}m_c\ddot{w}_t = 0 \tag{4}$$

$$u_{0b} - u_{0t} - (c/G_c)\tau + \left(c^3/12E_c\right)\tau_{,xx} + [(c+d_t)/2]w_{t,x} + [(c+d_b)/2]w_{b,x} = 0 \tag{5}$$

The boundary conditions for the upper and lower face sheets as well as for the core at the left ($x = 0$) and the right ($x = L$) edges of the sandwich beam are, for the upper face sheet,

$$EA_t u_{0t,x} = 0 \quad \text{or} \quad u_{0t} = 0 \tag{6}$$

$$EI_t w_{t,xx} = 0 \quad \text{or} \quad w_{t,x} = 0 \tag{7}$$

$$EI_t w_{t,xxx} - \tfrac{1}{2}bd_t\tau + \tfrac{1}{6}m_c d_t\ddot{u}_{0t} + \tfrac{1}{12}m_c d_t\ddot{u}_{0b} - \tfrac{1}{12}m_c d_t^2\ddot{w}_{t,x} + \tfrac{1}{24}m_c d_t d_b \ddot{w}_{b,x} = 0 \tag{8}$$

for the lower face sheet,

$$EA_b u_{0b,x} = 0 \quad \text{or} \quad u_{0b} = 0 \tag{9}$$

$$EI_b w_{b,xx} = 0 \quad \text{or} \quad w_{b,x} = 0 \tag{10}$$

$$EI_b w_{b,xxx} - \tfrac{1}{2}bd_b\tau - \tfrac{1}{12}m_c d_b\ddot{u}_{0t} - \tfrac{1}{6}m_c d_b\ddot{u}_{0b} - \tfrac{1}{12}m_c d_b^2\ddot{w}_{b,x} + \tfrac{1}{24}m_c d_t d_b \ddot{w}_{t,x} = 0 \tag{11}$$

and for the core,

$$\tau = 0 \quad \text{or} \quad w_c(x_c, z_c, t) = 0 \tag{12}$$

Finally, the vertical normal stresses in the core, as well as its vertical and longitudinal displacements in terms of the main unknowns (i.e., longitudinal and vertical face sheet displacements and shear stress in the core, respectively) are[4]

$$\sigma_{zz}(x_c, z_c, t) = \frac{E_c}{c}(w_b - w_t) + \left(\frac{c}{2} - z_c\right)\tau_{,x} \tag{13}$$

$$w_c(x_c, z_c, t) = -\frac{z_c(z_c - c)}{2E_c}\tau_{,x} + \frac{z_c}{c}(w_b - w_t) + w_t \tag{14}$$

$$u_c(x_c, z_c, t) = \frac{z_c}{G_c}\tau - \frac{z_c^2(3c - 2z_c)}{12E_c}\tau_{,xx} - \frac{z_c^2}{2c}w_{b,x} + \left(\frac{z_c^2}{2c} - z_c - \frac{d_t}{2}\right)w_{t,x} + u_{0t} \tag{15}$$

B. Finite Difference Discretization

The harmonic free vibration response of a sandwich beam in terms of the main unknown functions can be represented as[17]

$$\boldsymbol{u}(x, t) = \boldsymbol{u}(x)e^{i\omega t} \tag{16}$$

A central difference scheme with fictitious grid points at the boundaries has been developed so that the equations of motion are also satisfied at the beam edges, which is essential for obtaining the accurate numerical results in the case of sandwich structures with a soft core. Moreover, to increase the accuracy of the finite difference analysis, the highest order of the derivatives entering the mathematical formulation is reduced to two through the introduction of the new functions

$$\hat{w}_t(x) = \bar{w}_{t,xx}(x), \qquad \hat{w}_b(x) = \bar{w}_{b,xx}(x) \tag{17}$$

This leads to the generalized eigenvalue problem in the form

$$\boldsymbol{Kv} = \omega^2\boldsymbol{Mv} \tag{18}$$

Note that there are seven unknowns at each point of the finite difference grid. This eigenvalue problem with the large sparse matrices of the general type can be efficiently solved using the deflated iterative Arnoldi algorithm (see Ref. 18). A finite difference grid with a constant 0.5-mm interval between the grid points has been chosen for the higher-order numerical analysis of soft-core sandwich beams in the present paper. A grid refinement shows that the absolute value of the relative error in the fundamental frequency (smallest eigenvalue) and the highest calculated frequency (eighth eigenvalue) is only 3.5387*e*-5 and 2.4393*e*-4, respectively, with a grid twice as dense.

C. Confirmation of the Higher-Order Theory

The higher-order theory is verified here through comparison of the numerical simulations of the free vibration response obtained on its basis with published experimental results in the literature. The numerical analysis in this paper was carried out with the aid of the program FSAN developed in the MATLAB® software environment.[19]

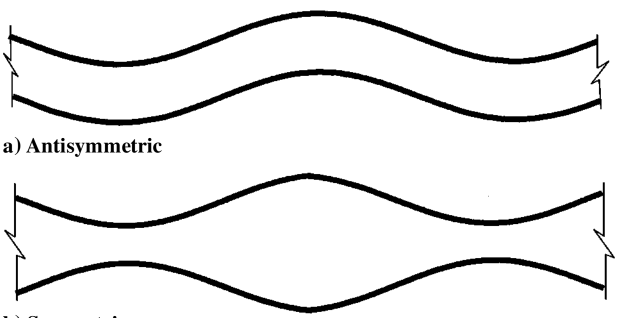
The first comparison is made with the television-holography measurements of sandwich vibrations reported by Jensen and Irgens.[16] They experimented with a simply supported sandwich beam having the following parameters. The span of the beam was 300 mm with a 10-mm free overhang at each end, the width of the beam was 50 mm, the identical face sheets were 2 mm thick, and the thickness

Table 1 Mechanical properties of the sandwich beam with a soft core

Component	Material	Young's modulus E, GPa	Shear modulus G, GPa	Poisson's ratio	Density ρ, kg/m^3
Face sheets	Steel	210	81	0.30	7900
Core	Divinycell H60	0.056	0.022	0.27	60

Table 2 Comparison of natural frequencies of a sandwich beam with a soft core

	Frequency, Hz			
	Antisymmetric modes		Symmetric modes	
Mode	Experiments[16]	HSAPT	Experiments[16]	HSAPT
1	263	245	——	2518
2	——	521	——	2526
3	889	856	——	2563
4	1289	1266	——	2661
5	1774	1762	——	2855
6	——	2352	3358	3179
7	——	3039	——	3649
8	3806	3826	——	4272
9	4621	4716	——	5041



a) Antisymmetric

b) Symmetric

Fig. 2 Vibration modes of sandwich beam.

of the soft plastic foam core was 30 mm. The mechanical properties of the materials are given in Table 1, and comparison between the predictions of the present theory and the measured vibrational modes is presented in Table 2. The antisymmetric mode in Table 2 involves a displacement pattern that is antisymmetric with respect to the beam midheight, that is, the face sheets move in phase with each other (Fig. 2a). Conversely, in the symmetric mode, the displacements are symmetric with respect to the beam midheight, that is, the face sheets move 180 deg out of phase with respect to each other (Fig. 2b).

As shown in Table 2, the greatest divergence between experiment and the present theory (−6.84%) occurs for the fundamental frequency. The reason for the discrepancy arises because the specimen was slightly clamped at each support to prevent it from sliding out of the supports, as reported in Ref. 16. Indeed, for the higher antisymmetric modes, the influence of this partial clamping diminishes, and the theoretical predictions and the experiment are much closer to each other. For the reasons discussed in Ref. 16, detection of the symmetric modes presented significant difficulties, and the experimenters were able to detect only a single frequency. This may explain the relatively high (−5.33%) discrepancy between theory and experiment in this case, because the analytically predicted frequencies for the nearest antisymmetric vibration modes enclosing this symmetric mode are in close agreement with the experiment measurement.

Note that the HSAPT produces the vibration frequencies and their corresponding modes in the natural ascending order of the frequencies, whereas the theoretical model described in Ref. 16 computes them independently. Moreover, the general character of the numerical analysis based on the HSAPT automatically predicts the appropriate vibrational mode, be it the shear or so-called thickness stretch mode (see Chapter 3 of Ref. 16).

The next comparison is made with the experimental results for a fixed–fixed sandwich beam with a stiff orthotropic core, as reported by Raville et al.[8] The span of the beam was 1219.2 mm long, the identical face sheets were 0.4064 mm thick, and the thickness of the core was 6.35 mm. The mechanical properties of the constituent materials are given in Table 3. (These properties and the experimental results of Raville et al.[8] can been found in Ref. 15.) Notice that the incompressibility of the core in the vertical direction assumed in Ref. 8 was simulated here by equating the Young's modulus of the core with that of the face sheets, that is, $E_c = E_t$.

Table 3 Mechanical properties of the sandwich beam with a stiff core

Component	Material	Young's modulus E, GPa	Shear modulus G, GPa	Poisson's ratio	Density ρ, kg/m^3
Face sheets	Aluminium	69	——	0.33	2687.3
Core	Paper	——	0.069	——	119.69

Table 4 Comparison of natural frequencies of a sandwich beam with a stiff core

	Frequency, Hz	
Mode No.	Experiments[8]	HSAPT
1	——	34.6
2	——	93.1
3	185.5	177.2
4	280.3	282.8
5	399.4	406.3
6	535.2	544.3
7	680.7	693.7
8	867.2	852.0
9	1020.0	1017.1
10	1201.0	1187.3

The theoretical predictions shown in Table 4 show the largest variance with the experimental values measured for the third mode (−4.5%). This is attributed to the deficiency reported in Ref. 8, namely, that the vibration exciter used in the experiments was not capable of generating a forcing function of the proper magnitude and frequency for the lower modes (modes 1–5).

The preceding comparisons of predicted modal frequencies with the experimental data indicate that the HSAPT is capable of accurate prediction of the natural vibrations for both stiff- and soft-core sandwich beams.

III. Numerical Study and Discussion

In this section, the higher-order analysis is used to study free vibration response of sandwich beams with a transversely flexible core. First, natural frequencies and corresponding natural modes of vibrations of the sandwich beams with different boundary conditions are determined, and distinctions between the vibration responses of homogeneous Euler–Bernoulli-type beams and sandwich beams are indicated. Next, the influence of increasing face sheet thickness, keeping the volume of the soft core constant, on the natural motions of the cantilever sandwich beam is studied. Finally, the effect of changing the volume of the soft core, with the thickness of the face sheets constant, is presented.

Free vibrations of sandwich beams for different supporting cases are studied assuming a single configuration and set of material properties, as shown in Fig. 3a. The mechanical properties of the face sheets correspond to an isotropic glass–ceramic of density 4400 kg/m^3, whereas those of the core correspond to isotropic polymethacrylimide rigid foam of density 52.060 kg/m^3. The first eight natural frequencies and corresponding natural modes are calculated for each supporting case.

Notice, in particular, that sandwich structures comprise skins that are much thinner than the core. For such thin skins, flexural stiffness is very small. Consequently, in the cases where clamped boundary conditions appear, the clamping effect is manifest only in the close vicinity of the clamped end. This distance is negligible compared with the beam span, making it impractical to portray a zero slope that could be discerned by the reader. For the case of relatively thick skins, however, the clamping effect is much more pronounced and could be visually observed.

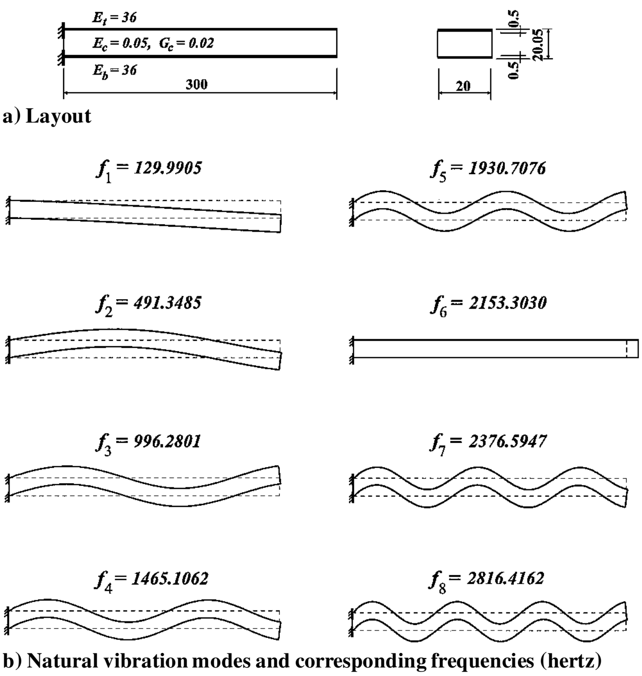


Fig. 3 Cantilever sandwich beam: case *A* in Fig. 8 (dimensions in millimeters; moduli in gigapascal).

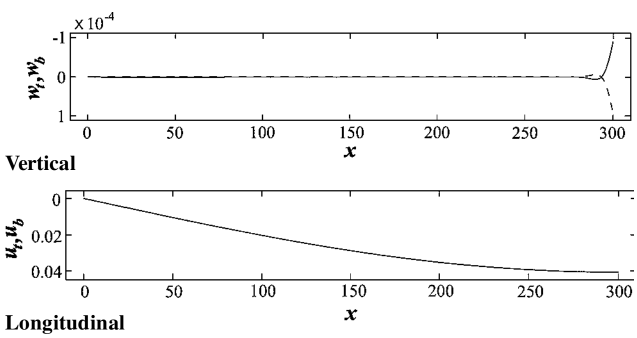


Fig. 4 Displacement patterns of cantilever sandwich beam for the sixth mode (——, top and - - -, bottom).

Natural motions of a cantilever sandwich beam are presented in Fig. 3. The right edge of the beam is free, and the ends of the face sheets are clamped at the left edge. The general character of the response resembles homogeneous beam behavior insofar as the number of nodal points increases with the natural frequency. However, unlike the homogeneous beam, a localized vibration mode appears between the fifth and seventh modes (Fig. 3b). The sixth mode consists predominantly of a longitudinal displacement pattern, although much smaller vertical displacements of the upper and lower face sheets also appear near the free end of the beam. These small vertical displacements are symmetric with respect to beam midheight (Fig. 4). In contrast, the dominating vertical displacements of the face sheets in all other modes are identical in magnitude and direction (antisymmetric with respect to the beam midheight), whereas the small longitudinal displacements are equal and oppositely directed. This behavior is a consequence of the symmetry of the sandwich beam layout about the midheight. Moreover, it can be observed from Fig. 3b that the vibration patterns of the sandwich beam are more complex than those of the homogeneous beam, with the difference being more pronounced for the lower modes. This is a direct consequence of the low shear modulus and compression modulus of the soft core. Notice that these effects are detectable because unlike other higher-order theories, the HSAPT accounts for vertical normal deformations and stresses (peeling stresses) in the soft core.

Free vibrations of a sandwich beam simply supported at both upper and lower face sheets and with free core edges are given in Fig. 5. The beam exhibits an axial vibration mode (sixth mode in Fig. 5b), instead of the localized one in the preceding case because of the restriction on the vertical movement at the right end. The value of the natural frequency corresponding to the longitudinal vibration mode is essentially the same as in the case of the cantilever beam (compare Figs. 3b and 5b), and the negligible increment in the natural frequency is accounted for by the stiffer support conditions. The natural frequencies corresponding to the other modes also increase, respectively, as compared with Fig. 3b. Like the cantilever beam, the face sheets also move antisymmetrically in the vertical direction for all modes, except for the sixth one.

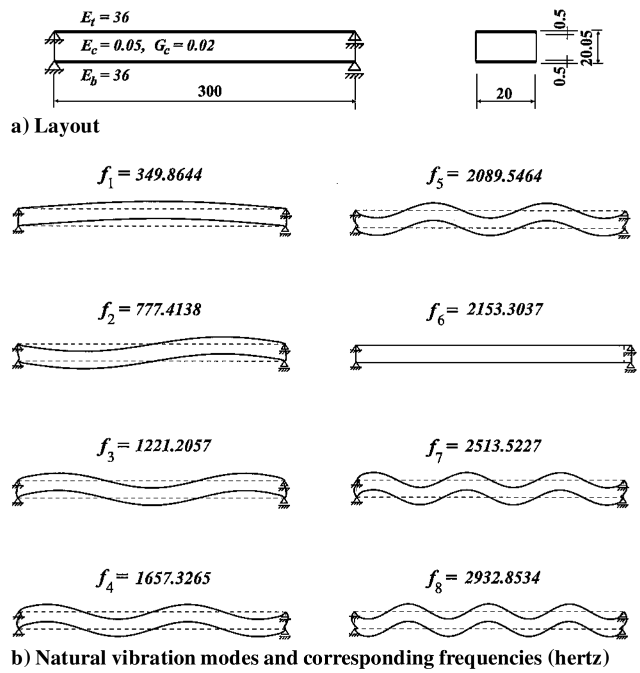


Fig. 5 Sandwich beam simply supported at both upper and lower face sheets (dimensions in millimeters; moduli in gigapascal).

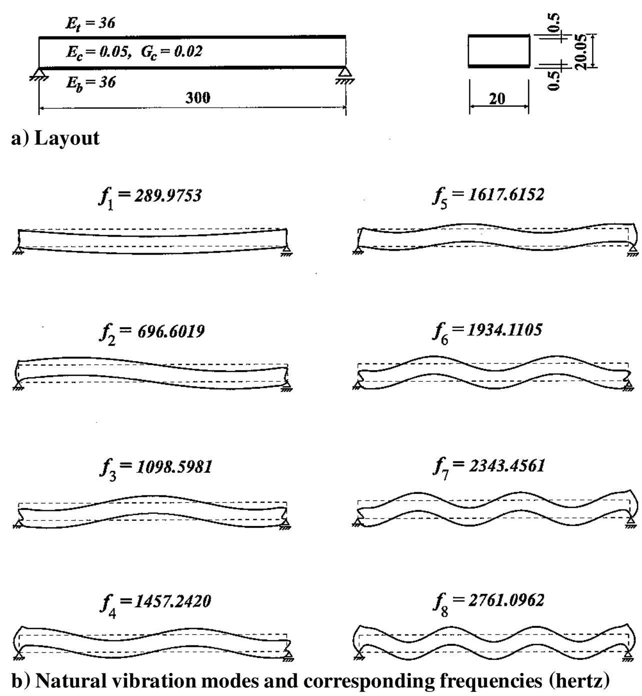


Fig. 6 Sandwich beam simply supported at lower face sheet only (dimensions in millimeters; moduli in gigapascal).

Figure 6a shows a sandwich beam that is simply supported only at the lower face sheet. Because of the lack of midheight symmetry in this case, the axial vibration mode encountered in the preceding example does not exist. Instead, a fifth mode is predicted in which the horizontal and vertical displacements of the sandwich beam are of the same order of magnitude. In other vibration modes, the vertical

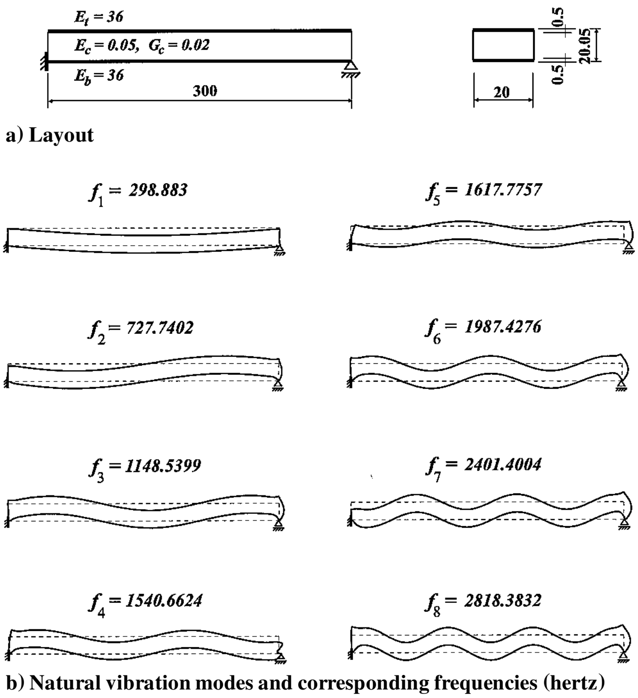


Fig. 7 Sandwich beam with boundary conditions of general type (dimensions in millimeters; moduli in gigapascal).

displacements dominate. Another important consequence of the unsupported upper face sheet is that different vertical displacements of the face sheets at the beam edges are predicted for all modes (Fig. 6b).

Natural motions of a generally supported sandwich beam are shown in Fig. 7. This configuration is obtained from the layout of the preceding case by changing the left pin to a clamped support. The substitution results in a minor increase of the natural frequencies and local changes in the corresponding vibration modes in the vicinity of the left support (compare to Fig. 6). Furthermore, Figs. 6b and 7b show that the fifth vibration mode is nearly identical for the two cases. Because the left edge is only clamped locally (at the lower face sheet, Fig. 7a) and because the fifth mode vibrations take place in both the horizontal and vertical directions the effect of clamping (as compared to pinning) is reduced.

Notice that, when the upper and lower face sheets of the sandwich are clamped and the edges of the transversely flexible core are free of traction, the character of the vibration response closely resembles that of a homogeneous beam. That is a direct consequence of precluding the possibility of horizontal movement. However, the natural frequencies of the sandwich beam are closer to each other than those of a homogeneous beam.

The present numerical analysis based on the higher-order theory is equally capable of treating sandwich beams with an asymmetric section. However, the main qualitative features of the vibration response of the soft-core sandwich beams are more readily perceived using beams with a symmetric section. The primary distinctive characteristics of the vibration response of sandwich beams with nonidentical face sheets as compared to those considered already are the disruption of the antisymmetric pattern for the higher vibration modes and the less pronounced localized and axial vibration effects.

The effect of increasing face sheet thickness on the vibration response of a cantilever sandwich beam appears in Fig. 8. (The core volume and layout is otherwise the same as in Fig. 3a.) Figure 8 shows variation of natural logarithm of the nondimensional frequency parameter $\bar{\omega} = \omega/\sqrt{[E_t c/(\rho_t d L^2)]}$ for the first eight natural frequencies of the cantilever sandwich beam as the thickness of each face sheet is increased from 0.2 to 5.6 mm. The solid frequency curves correspond to the antisymmetric vibration modes, whereas the dashed curve corresponds to the localized vibration mode (see mode 6 in Fig. 3b). For example, point A on the plot corresponds to the sandwich layout of Fig. 3a, with a face sheet thickness of 0.5 mm ($h/c = 1.0525$) and a localized sixth mode (in ascending order).

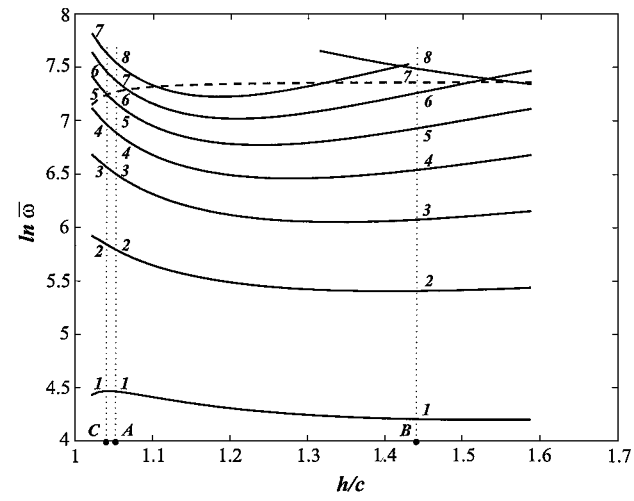


Fig. 8 Dependence of nondimensional frequency parameter $\bar{\omega}$ on sandwich thickness ratio h/c for cantilever sandwich beam with constant volume of soft core.

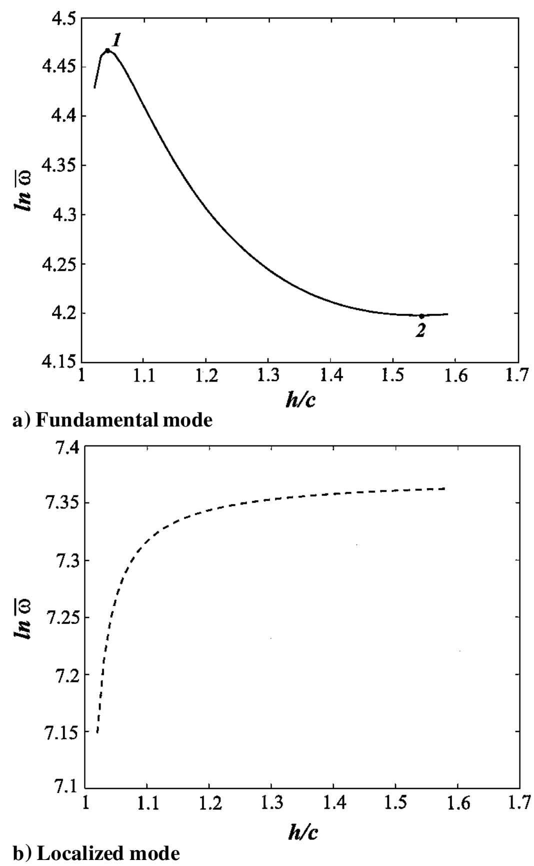
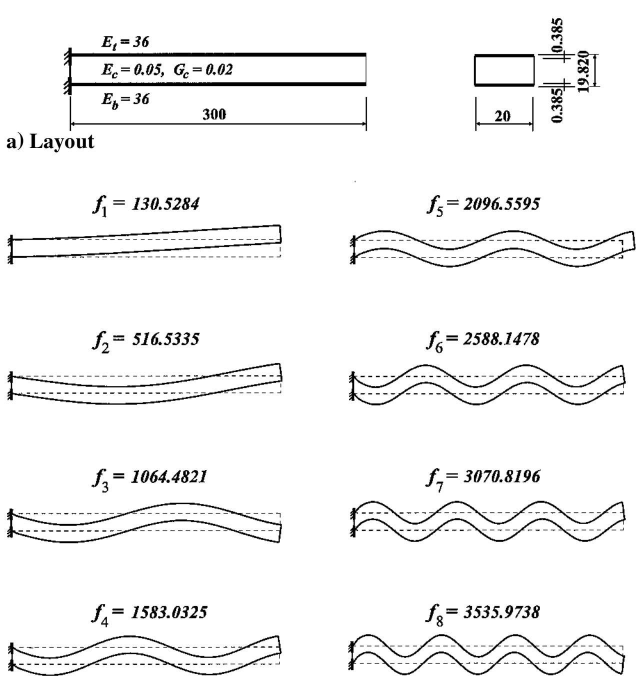


Fig. 9 Magnified view of selected modes from Fig. 8.

A number of important observations concerning natural motions of sandwich beams follow from the plot of Fig. 8. The first observation concerns the fundamental frequency of the sandwich beam. For clarity, the curve representing variation of the fundamental frequency with the sandwich thickness ratio (h/c) is given separately in Fig. 9a. Figures 8 and 9a show that the fundamental frequency of the cantilever sandwich beam possesses a local maximum at point 1, $h/c = 1.042$ ($d_t = 0.4$ mm) and a local minimum at point 2, $h/c = 1.5459$ ($d_t = 5.2$ mm). This means that the fundamental frequency of the cantilever sandwich beam briefly increases as the face sheet thickness is increased, followed by a steady decline until the face sheets become relatively thick. (This phenomenon was mentioned in Ref. 10 regarding sandwich beams with a stiff core.) The other solid frequency curves, except for the short descending



b) Natural vibration modes and corresponding frequencies (hertz)

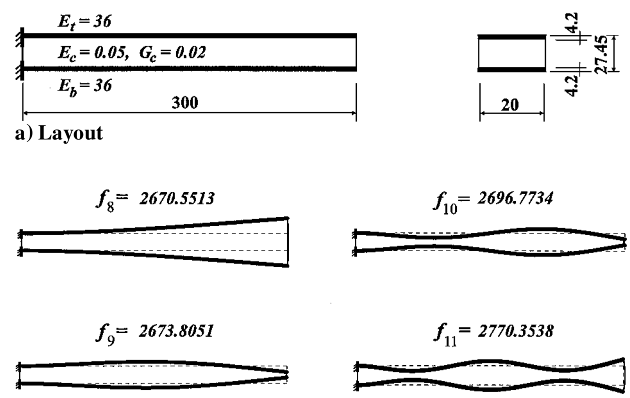
Fig. 10 Interactive mode vibration response: case *C* in Fig. 8 (dimensions in millimeters; moduli in gigapascal).

one discussed hereafter, do not possess local maxima and decrease monotonically until they reach their local minimum points. The higher the particular natural frequency, the lower the ratio (h/c) for which the corresponding frequency curve reaches its local minimum point and then starts to grow.

Furthermore, the dashed frequency curve in Fig. 8 corresponding to the localized vibration mode does not possess an extremum point. As shown in Fig. 9b, the curve rises steeply in the interval $h/c \sim 1.021$–1.15, then continues to grow, albeit more slowly. The intersection of the frequency curve corresponding to the localized vibration mode with the solid frequency curves corresponding to antisymmetric vibration modes implies that there are cantilever sandwich beam layouts for which the localized frequency mode alone does not exist (Fig. 10). Instead, it interacts with the adjacent antisymmetric vibration mode (mode 5 in Fig. 3b) to produce an interactive mode vibration response, shown as the fifth mode in Fig. 10b. The beam configuration of Fig. 10 (corresponding to point C in Fig. 8) has been obtained with the aid of the bisection procedure in conjunction with the higher-order free vibration analysis under consideration.

Because for homogeneous beams natural frequency increases with beam stiffness, one could conclude that higher vibration modes of sandwich beams are qualitatively closer to the corresponding modes of homogeneous beams. For the sandwich beam, however, an increase in stiffness caused by an increase in face sheet thickness leads to a reduction in natural frequencies, at least for a limited range of face sheet thicknesses. To understand this phenomenon, the interaction between the face sheets and the transversely flexible core should be taken into consideration, apart from the usual criteria. This interaction plays a critical role in the sandwich response and depends on the value of the sandwich thickness ratio (h/c). Thus, the local minimum on the particular curve of the natural frequency can be interpreted as the end of the sandwich beam response zone and the start of the homogeneous beamlike response zone for the corresponding vibration mode.

The influence of the thickness ratio h/c on the interaction between the face sheets and the core is clearly demonstrated for thick face sheets. Figure 11a presents a sandwich beam with relatively thick (4.2 mm) face sheets $(h/c = 1.4409)$, corresponding to point B in Fig. 8. It follows from Fig. 8 that the first five vibration modes of the configuration considered are qualitatively the same as those of case A, whereas its sixth and seventh modes qualitatively correspond to the seventh and sixth modes of case A, respectively (Fig. 3b).



b) Symmetric vibration modes and corresponding frequencies (hertz)

Fig. 11 Sandwich beam with relatively thick face sheets: case *B* in Fig. 8 (dimensions in millimeters; moduli in gigapascal).

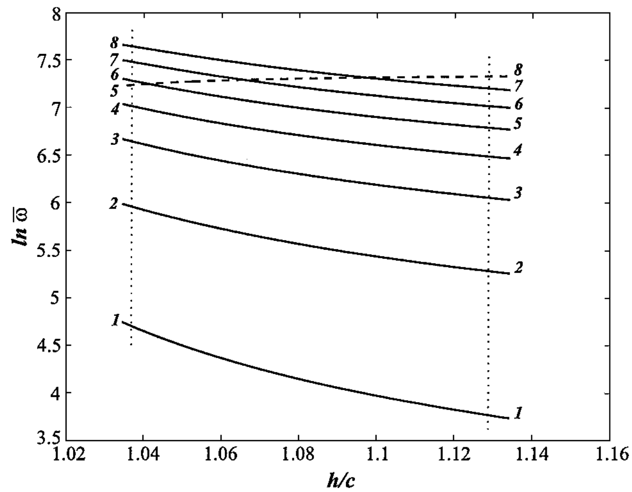


Fig. 12 Dependence of nondimensional frequency parameter $\bar{\omega}$ on sandwich thickness ratio *h/c* for cantilever sandwich beam with invariant thickness of face sheets.

Furthermore, in the present case, the response at the higher frequencies differs significantly from the response of the sandwich beam with thin face sheets. Namely, the antisymmetric vibration modes, which follow the usual pattern of increasing the nodal points, are shifted to higher frequencies, giving way to symmetric vibration modes. Figure 11b demonstrates that the thick face sheets vibrate in a local manner (i.e., independently of each other, at modes 8–11), following the usual pattern of increasing the nodal points, now in the local sense, as opposed to the case of antisymmetric modes, where vibration takes place in a global sense. Notice, in particular, the close proximity of the symmetric vibration modes to each other (also see Ref. 16).

The short descending solid frequency curve in Fig. 8 corresponds to the eighth vibration mode of Fig. 11b. (It is noted, for the sake of clarity, that the left starting portion of this curve corresponds to the natural frequencies somewhat higher than the eighth one.) The greater the thickness of the face sheets, the lower the natural frequencies corresponding to the symmetric vibration modes.

The effect of changing the soft core volume with constant face sheet thickness ($d_t = 0.5$ mm), is shown in Fig. 12, where the core thickness varies from $h/c = 1.0345$–1.1342 ($c = 29.95$–7.45 mm). As shown in Fig. 12, the solid frequency curves corresponding to the antisymmetric vibration modes do not possess any extremum points, and the natural frequencies are higher for the thick-core beams than for the thin-core ones. However, the dashed frequency curve corresponding to the localized vibration mode exhibits the opposite behavior. The thinner soft core contributes less to the vibration energy that derives from the small vertical component of motion, and the part of the vibration energy corresponding to the axial motion of the stiffer face sheets increases.

IV. Conclusions

1) Comparisons with published experimental data demonstrate the ability of the HSAPT to predict accurately the natural vibration frequencies of sandwich beams with both soft (foam) and stiff (honeycomb) cores under various supporting conditions. The higher-order frequencies are predicted with the same accuracy as the lower ones, a task that constitutes a significant challenge for other analytical approaches, including that of finite elements.

2) The vibration modes of a sandwich beam with thin face sheets and constrained longitudinal displacements, which correspond to the low and moderately high natural frequencies, consist predominantly of antisymmetric vertical displacements of the face sheets. For a cantilever sandwich beam, there is a localized vibration pattern consisting of the longitudinal displacements along the major portion of the span and small symmetrical vertical displacements in the vicinity of the free end.

3) When the face sheets become relatively thick, the higher antisymmetric vibration modes having a global character shift to higher frequencies, giving way to symmetric vibration modes comprising local vibrations of the face sheets.

4) The face sheet deflections of a sandwich beam with nonidentical support conditions at the same boundary section are different in the close vicinity of that section.

5) The natural frequencies of the sandwich beams with face sheets of different thickness and constant volume soft cores decrease with increasing sandwich stiffness through some interval of variation of the sandwich thickness ratio. The increase in the natural frequencies of the sandwich beam begins only after passing a threshold value of the sandwich thickness ratio. This value is different for each antisymmetric mode and is smaller for higher modes and larger for lower ones.

6) The increase/decrease in core volume for face sheets of constant thickness leads to an increase/decrease of the natural frequencies of the antisymmetric vibration modes and to a decrease/increase of the natural frequency corresponding to the localized vibration mode of a cantilever sandwich beam.

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A. Berman
Associate Editor